

SOLITON-LIKE SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE QUADRATIC HAMILTONIANS

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ABSTRACT. We construct one soliton solutions for the nonlinear Schrödinger equation with variable quadratic Hamiltonians in a unified form by taking advantage of a complete (super) integrability of generalized harmonic oscillators. The soliton wave evolution in external fields with variable quadratic potentials is totally determined by the linear problem, like motion of a classical particle with acceleration, and the (self-similar) soliton shape is due to a subtle balance between the linear Hamiltonian (dispersion and potential) and nonlinearity in the Schrödinger equation by the standards of soliton theory. Most linear (hypergeometric, Bessel) and a few nonlinear (Jacobian elliptic, second Painlevé transcendental) classical special functions of mathematical physics are linked together through these solutions, thus providing a variety of nonlinear integrable cases. Examples include bright and dark solitons, and Jacobi elliptic and second Painlevé transcendental solutions for several variable Hamiltonians that are important for current research in nonlinear optics and Bose–Einstein condensation. The Feshbach resonance matter wave soliton management is briefly discussed from this new perspective.

1. INTRODUCTION

Advances of the past decades in nonlinear optics, Bose–Einstein condensates, propagation of soliton waves in plasma physics and in other fields of nonlinear science have involved a detailed study of nonlinear Schrödinger equations (see, for example, [9], [17], [81], [149], [151] and references therein). In the theory of Bose–Einstein condensation [42], [114], from a general point of view, the dynamics of gases of cooled atoms in a magnetic trap at very low temperatures can be described by an effective equation for the condensate wave function known as the Gross–Pitaevskii (or nonlinear Schrödinger) equation [16], [69], [70], [76], [108] and [113]. Experimental observations of dark and bright solitons [18], [20], [44], [75] and bright soliton trains [9], [129], [130] in the presence of harmonic confinement have generated considerable research interest in this area [16], [58].

The propagation of an optical pulse in a real fiber is also well described by a nonlinear Schrödinger equation for the envelope of wave functions travelling inside the fiber [4], [7], [17], [46], [61], [77]. A class of self-similar solutions that exists for physically realistic dispersion and nonlinearity profiles in a fiber with anomalous group velocity dispersion is discussed in [78], [79], [97], [98], [116], [123], [125], which suggests, among other things, a method of pulse compression and a model of steady-state asynchronous laser mode locking [98]. Solutions of a nonhomogeneous Schrödinger equation are also known for propagation of soliton waves in plasma physics [13], [22], [23], [104].

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Integration techniques of the nonlinear Schrödinger equation include Painlevé analysis [10], [17], [30], [31], [32], [33], [63], [81], [102], [144], Hirota method [64], [65], [81], Lax method [10], [81], [84], [151], Miura transformation [94], [95], inverse scattering transform and Hamiltonian approach [2], [3], [6], [55], [59], [105] among others [21], [45], [51], [89], [107], [120]. Although the classical soliton concept was developed for nonlinear autonomous dispersive systems with time being an independent variable only, not appearing in the nonlinear evolution equations (see [126], [127] for highlighting this point), connections between autonomous and nonautonomous Schrödinger equations have been discussed in [2], [4], [27], [62], [82], [98], [112], [116] and [155] (see Remark 2 for an explicit transformation). The formation of matter wave solitons in Bose–Einstein condensation by magnetically tuning the interatomic interaction near the Feshbach resonance provides an example of nonautonomous systems that are currently under investigation [16], [58], [130].

We elaborate on results of recent papers [9], [10], [12], [17], [52], [85], [24], [62], [63], [78], [79], [82], [80], [118], [123], [124], [125], [126], [127], [129], [140], [146], [147], [148], [149] on construction of exact solutions of the nonlinear Schrödinger equation with variable quadratic Hamiltonians (see also [136] and [151], [152], [153]). In this paper, a unified form of these soliton-like (self-similar) solutions is presented, thus combining progress of the soliton theory with a complete integrability of generalized harmonic oscillators. We show, in general, that the soliton evolution in external fields described by variable quadratic potentials is totally determined by the linear problem, similar to the motion of a classical particle with acceleration, while the original soliton shape is due to a delicate balance between the linear Hamiltonian (dispersion and potential) and nonlinearity in the Schrödinger equation according to basic principles of the soliton theory. Examples include bright and dark solitons, and Jacobi elliptic and Painlevé II transcendental solutions for solitary wave profiles, which are important in nonlinear optics [4], [22], [23], [78], [79], [97], [123], [125], [151] and Bose–Einstein condensation [12], [9], [127], [129], [148].

The paper is organized as follows. We present a unified form of one soliton solutions with integrability conditions, and sketch the proof in the next two sections, respectively. In Section 4, more details are provided and some simple examples are discussed. Section 5 deals with a Feshbach resonance management of matter wave solitons. In the last section, an extension of our method is given and a classical example of accelerating soliton in a linearly inhomogeneous plasma [22], [23] is revisited from a new perspective. An attempt to collect most relevant bibliography is made but in view of a rich history ¹ and the very high publication rate in these research areas we must apologize in advance if some important papers are missing.

2. SOLITON-LIKE SOLUTIONS

The nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi + g\psi + h|\psi|^2\psi, \quad (2.1)$$

where the variable Hamiltonian H is a quadratic form of operators $p = -i\partial/\partial x$ and x , namely,

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi + g(x,t)\psi + h(t)|\psi|^2\psi \quad (2.2)$$

¹Ref. [45] presents a detailed source on classical papers in the soliton theory.

(a, b, c, d are suitable real-valued functions of time only) has the following soliton-like solutions

$$\begin{aligned} \psi(x, t) &= \frac{e^{i\phi}}{\sqrt{\mu(t)}} \exp(i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2)) \\ &\times F(\beta(t)x + 2\gamma(t)y) \end{aligned} \quad (2.3)$$

(ϕ is a real constant, y is a parameter and $\mu, \alpha, \beta, \gamma$ are real-valued functions of time only given by equations (2.16)–(2.22) below), provided that

$$g = g_0 a(t) \beta^2(t) (\beta(t)x + 2\gamma(t)y)^m, \quad h = h_0 a(t) \beta^2(t) \mu(t) \quad (2.4)$$

(g_0 and h_0 are constants and $m = 0, 1$). As we shall see in the next section, these (integrability) conditions control the balance between the linear Hamiltonian (dispersion and potential) and non-linearity in the Schrödinger equation (2.2) thus making possible an existence of the soliton-like solution (with damping or amplification) in the presence of variable quadratic potentials (see also [12], [63], [126], [127] and [154] for discussion of important special cases; Remark 2 provides an important interpretation of relations (2.4) as a complete integrability condition for the nonautonomous nonlinear Schrödinger equation (2.1) when $m = 0$).

Here, the soliton profile function $F(z)$ of a single travelling wave-type argument $z = \beta x + 2\gamma y$ satisfies the ordinary nonlinear differential equation of the form

$$F''(z) = g_0 z^m F(z) + h_0 F^3(z). \quad (2.5)$$

If $m = 0$, with the help of an integrating factor,

$$\left(\frac{dF}{dz}\right)^2 = C_0 + g_0 F^2 + \frac{1}{2} h_0 F^4 \quad (C_0 \text{ is a constant}), \quad (2.6)$$

which can be solved in terms of Jacobian elliptic functions [8], [53], [81], [143]. When $m = 1$, equation (2.5) leads to Painlevé II transcendents [6], [28], [31], [33], [81].

The variable phase is given in terms of solutions of the following system of ordinary differential equations:

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = 0, \quad (2.7)$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \quad (2.8)$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \quad (2.9)$$

(see Ref. [34] and the next section for more details), where the standard substitution

$$\alpha = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)} \quad (2.10)$$

reduces the Riccati equation (2.7) to the second order linear equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \quad (2.11)$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left(\frac{a'}{a} - \frac{d'}{d} \right). \quad (2.12)$$

(Relations with the corresponding Ehrenfest theorem for the linear Hamiltonian are discussed in Ref. [36].)

It is worth noting that in the soliton-like solution under consideration (2.3) linear and nonlinear factors are essentially separated, namely, the nonlinear part is represented only by the profile function F of a single travelling wave variable $z = \beta x + 2\gamma y$ as solution of the nonlinear equation (2.5). Letting $\beta x + 2\gamma y = \text{constant}$, one obtains

$$x' + \frac{\beta'}{\beta}x = 2a\beta y \quad (2.13)$$

and

$$x'' - \frac{a'}{a}x' + \left(\left(\frac{\beta'}{\beta} \right)' - \frac{a'}{a} \frac{\beta'}{\beta} - \left(\frac{\beta'}{\beta} \right)^2 \right) x = 0 \quad (2.14)$$

for the soliton velocity and acceleration with the aid of (2.9). Then, by (2.7)–(2.8):

$$x'' - \frac{a'}{a}x' + \left(4ab - c^2 + c \left(\frac{a'}{a} - \frac{c'}{c} \right) \right) x = 0 \quad (2.15)$$

that is similar to equation of motion of a classical particle (damped parametric oscillations).

The initial value problem for the system (2.7)–(2.9), which corresponds to the linear Schrödinger equation with a variable quadratic Hamiltonian (generalized harmonic oscillators [15], [49], [60], [145], [150]), can be explicitly solved in terms of solutions of our characteristic equation (2.11) as follows [34], [36], [132], [133]:

$$\mu(t) = 2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t)), \quad (2.16)$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \quad (2.17)$$

$$\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\mu(0)}{\mu(t)}\lambda(t), \quad (2.18)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))}, \quad (2.19)$$

where

$$\alpha_0(t) = \frac{1}{4a(t)} \frac{\mu'_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \quad (2.20)$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \quad \lambda(t) = \exp \left(- \int_0^t (c(s) - 2d(s)) ds \right), \quad (2.21)$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)} \quad (2.22)$$

provided that μ_0 and μ_1 are the standard solutions of equation (2.11) corresponding to the following initial conditions $\mu_0(0) = 0$, $\mu'_0(0) = 2a(0) \neq 0$ and $\mu_1(0) \neq 0$, $\mu'_1(0) = 0$. (Formulas (2.20)–(2.22) correspond to Green's function of generalized harmonic oscillators; see, for example, [34], [36], [50], [88], [132], [133] and references therein for more details.)

The continuity with respect to initial data,

$$\lim_{t \rightarrow 0^+} \alpha(t) = \alpha(0), \quad \lim_{t \rightarrow 0^+} \beta(t) = \beta(0), \quad \lim_{t \rightarrow 0^+} \gamma(t) = \gamma(0), \quad (2.23)$$

has been established in [132] for suitable smooth coefficients of the linear Schrödinger equation. Thus the soliton-like solution (2.3) evolves to the future $t > 0$ starting from the following initial data:

$$\begin{aligned}\psi(x, 0) &= \lim_{t \rightarrow 0^+} \psi(x, t) \\ &= \frac{e^{i\phi}}{\sqrt{\mu(0)}} \exp(i(\alpha(0)x^2 + \beta(0)xy + \gamma(0)y^2)) \\ &\quad \times F(\beta(0)x + 2\gamma(0)y),\end{aligned}\tag{2.24}$$

where $\phi, \mu(0), \alpha(0), \beta(0), \gamma(0)$ and y are arbitrary real parameters (see also (6.21) for a more general solution of this form).

Remark 1. When $m = 0$, the gauge transformation $\psi = \chi(x, t) \exp[ig_0(\gamma(t) - \gamma(0))]$ changes the original equation (2.2) into

$$i\chi_t = -a(t)\chi_{xx} + b(t)x^2\chi - ic(t)x\chi_x - id(t)\chi + h(t)|\chi|^2\chi,\tag{2.25}$$

where a, b, c, d are suitable real-valued functions of time t only and

$$h(t) = h_0\beta^2(0)\mu^2(0)\frac{a(t)\lambda^2(t)}{\mu(t)},\tag{2.26}$$

which is more common in practice. Once again, classical solution of the linear equation (2.11), namely, our characteristic function $\mu(t)$, completely controls the specific form of the nonlinearity factor $h(t)$ required for creation of the soliton (an extension is given in Section 6; see also Refs. [12], [126] and [127] for important special cases).

Remark 2. A simple change of variables,

$$\psi = \frac{1}{\sqrt{\mu}} e^{i\alpha x^2} \chi(\beta x, \gamma),\tag{2.27}$$

transforms the nonautonomous equation (2.2) with conditions (2.4), when $m = 0$, into a standard autonomous nonlinear Schrödinger equation with respect to the new variables $\xi = \beta x$ and $\tau = \gamma$:

$$i\chi_t + g_0\chi + h_0|\chi|^2\chi = \chi_{\xi\xi},\tag{2.28}$$

which is completely integrable by advanced methods of the soliton theory [2], [6], [81], [151] (see also [45] and references cited in the introduction). This observation provides an alternative approach to derivation of our equations (2.7)–(2.10). An extension of the transformation (2.27) is given in [134].

3. SKETCH OF THE PROOF

Following [34] (see also [22], [91] and [118]), we are looking for exact solutions of the form

$$\psi = A(x, t) e^{iS(x, t)}, \quad S(x, t) = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2\tag{3.1}$$

(y is a parameter). Substituting into (2.2) and taking the imaginary part,

$$A_t + ((4a\alpha + c)x + 2a\beta y)A_x + (2\alpha a + d)A = 0.\tag{3.2}$$

For the real part, equating coefficients of all admissible powers of $x^m y^n$ with $m + n = 2$, one gets our system of ordinary differential equations (2.7)–(2.9) of the corresponding linear Schrödinger

equation with the unique solution (2.16)–(2.22) already obtained in Refs. [34], [132], [133] and/or elsewhere. In addition, an auxiliary nonlinear equation of the form

$$aA_{xx} = gA + hA^3 \quad (3.3)$$

appears as a contribution from the last two terms. With the help of (2.8) and (2.10) our equation (3.2) can be rewritten as

$$A_t - \left(\frac{\beta'}{\beta}x - 2a\beta y \right) A_x + \frac{1}{2} \frac{\mu'}{\mu} A = 0. \quad (3.4)$$

Looking for a travelling wave solution with damping or amplification:

$$A = A(x, t) = \frac{1}{\sqrt{\mu(t)}} F(z), \quad z = c_0(t)x + c_1(t)y, \quad (3.5)$$

one gets

$$c'_0x + c'_1y = \left(\frac{\beta'}{\beta}x - 2a\beta y \right) c_0 \quad (3.6)$$

with $c_0 = \beta$ and $c_1 = 2\gamma$ (or $z = \beta x + 2\gamma y$). Then equation (3.3) takes the form

$$\frac{d^2}{dz^2} F(z) = \frac{g}{a\beta^2} F(z) + \frac{h}{a\beta^2\mu} F^3(z), \quad (3.7)$$

which must have all coefficients depending on z only in order to preserve a self-similar profile of the travelling wave with damping or amplification. This results in the required equation (2.5) under the balancing conditions (2.4) and our proof is complete. (An extension is given in Section 6.)

Remark 3. Assuming in (3.4) that

$$A(x, t) = \frac{1}{\sqrt{\mu(t)}} B(\xi, \tau), \quad (3.8)$$

where $\xi = \beta x$ and $\tau = 2\gamma y$, implies $B_\xi = B_\tau$ with a general solution $B(\xi, \tau) = F(\xi + \tau) = F(\beta x + 2\gamma y)$.

4. DETAILS AND EXAMPLES

A brief description of the method under consideration is as follows. In order to obtain soliton-like solutions (2.3) explicitly, say in terms of elementary and/or transcendental functions, one has to solve, in general, the nonlinear equation (2.5) for the profile function $F(z)$ in terms of Jacobian elliptic functions [8], [53], [81], [115], [143] (some elementary solutions are also available), when $m = 0$, or in terms of Painlevé II transcendents, when $m = 1$ (it is known that if $m > 1$, this equation does not have the Painlevé property [6], [81]). In addition, one has to solve the linear characteristic equation (2.11), which has a variety of solutions in terms of elementary and special (hypergeometric, Bessel) functions [11], [87], [103], [115], [142]. Many elementary solutions of the corresponding linear Schrödinger equation for generalized harmonic oscillators are known explicitly (see, for example, [34], [35], [36], [37], [50], [88], [133], [145], [150] and references therein). Then, the linear part allows determination of the travelling wave argument $z = \beta x + 2\gamma y$ and the damping (or amplifying) factor $\mu^{-1/2}$ of the soliton-like solution (2.3). Our balancing conditions (2.4) control dispersion, potential and nonlinearity in the original nonlinear Schrödinger equation (2.2), which is crucial for the soliton existence. (An extension is discussed in Section 6.)

4.1. Nonlinear Part. When $m = 0$, equation (2.5) is integrated to the first order equation (2.6) and (the corresponding initial value problem) can be solved by the reduction of elliptic integrals in terms of Jacobian or Weierstrass (doubly) periodic elliptic functions [8], [53], [79], [81]. We are interested in real-valued solutions. Some of the classical nonlinear wave configurations are given by

$$F(z) = \left(\frac{g_0 + \sqrt{g_0^2 - 2C_0 h_0}}{-h_0} \right)^{1/2} \times \text{cn} \left((g_0^2 - 2C_0 h_0)^{1/4} z, \left(\frac{g_0 + \sqrt{g_0^2 - 2C_0 h_0}}{2\sqrt{g_0^2 - 2C_0 h_0}} \right)^{1/2} \right), \quad (4.1)$$

if $h_0 < 0$ and

$$F(z) = \left(\frac{-g_0 + \sqrt{g_0^2 - 2C_0 h_0}}{h_0} \right)^{1/2} \times \text{sn} \left(\left(\frac{C_0 h_0}{-g_0 + \sqrt{g_0^2 - 2C_0 h_0}} \right) z, \left(\frac{g_0 - \sqrt{g_0^2 - 2C_0 h_0}}{g_0 + \sqrt{g_0^2 - 2C_0 h_0}} \right)^{1/2} \right), \quad (4.2)$$

if $g_0 < 0$. Here, $\text{cn}(u, k)$ and $\text{sn}(u, k)$ are the Jacobi elliptic functions [8], [53], [143]. Familiar special cases include the *bright* soliton:

$$F(z) = \sqrt{\frac{2g_0}{-h_0}} \frac{1}{\cosh(\sqrt{g_0}z)} \quad (4.3)$$

with $C_0 = 0$ in (4.1) and the *dark* soliton:

$$F(z) = \sqrt{\frac{-g_0}{h_0}} \tanh \left(\sqrt{\frac{-g_0}{2}} z \right) \quad (4.4)$$

with $C_0 = g_0^2/(2h_0)$ in (4.2), when $\text{cn}(u, 1) = 1/\cosh u$ and $\text{sn}(u, 1) = \tanh u$, respectively (the real period tends to infinity). More details can be found in Refs. [8], [53], [57], [79], [143], [151] and/or elsewhere.

If $m = 1$, the substitution $F(z) = g_0^{1/3} \sqrt{2/h_0} w(\zeta)$ and $\zeta = zg_0^{1/3}$ transforms (2.5) into the second Painlevé equation,

$$w'' = \zeta w + 2w^3. \quad (4.5)$$

In the limit $w \rightarrow 0$, this equation reduces to the Airy equation and its solution may be thought of as a nonlinear generalization of an Airy function [5]. There is a one-parameter family of real solutions $w = A_k(\zeta)$ that are bounded for all real ζ with the following asymptotic properties:

$$A_k(\zeta) = \begin{cases} k \text{Ai}(\zeta), & \zeta \rightarrow +\infty \\ r |\zeta|^{-1/4} \sin(s(\zeta) - \theta_0) + o(|\zeta|^{-1/4}), & \zeta \rightarrow -\infty \end{cases} \quad (4.6)$$

Here, $\text{Ai}(\zeta)$ is the Airy function, $-1 < k < 1$ provided $k \neq 0$, $r^2 = -\pi^{-1} \ln(1 - k^2)$,

$$s(\zeta) = \frac{2}{3} |\zeta|^{3/2} - \frac{3}{4} r^2 \ln |\zeta| \quad (4.7)$$

and

$$\theta_0 = \frac{3}{2}r^2 \ln 2 + \arg \Gamma \left(1 - \frac{i}{2}r^2 \right) + \frac{\pi}{4} (1 - 2\text{sign}(k)). \quad (4.8)$$

These asymptotics were found in [5], [122] and, eventually, had been proven rigorously in [47], [48] (see [6], [14], [28], [29], [31], [33], [135] and references therein for study of this nonlinear Airy function; graphs of these functions are presented in [28], see Figure 1 for an example: $w_{.5} = A_{1/2}(x)$.) An application to a soliton moving with a constant velocity in linearly inhomogeneous plasma is discussed in Section 6.

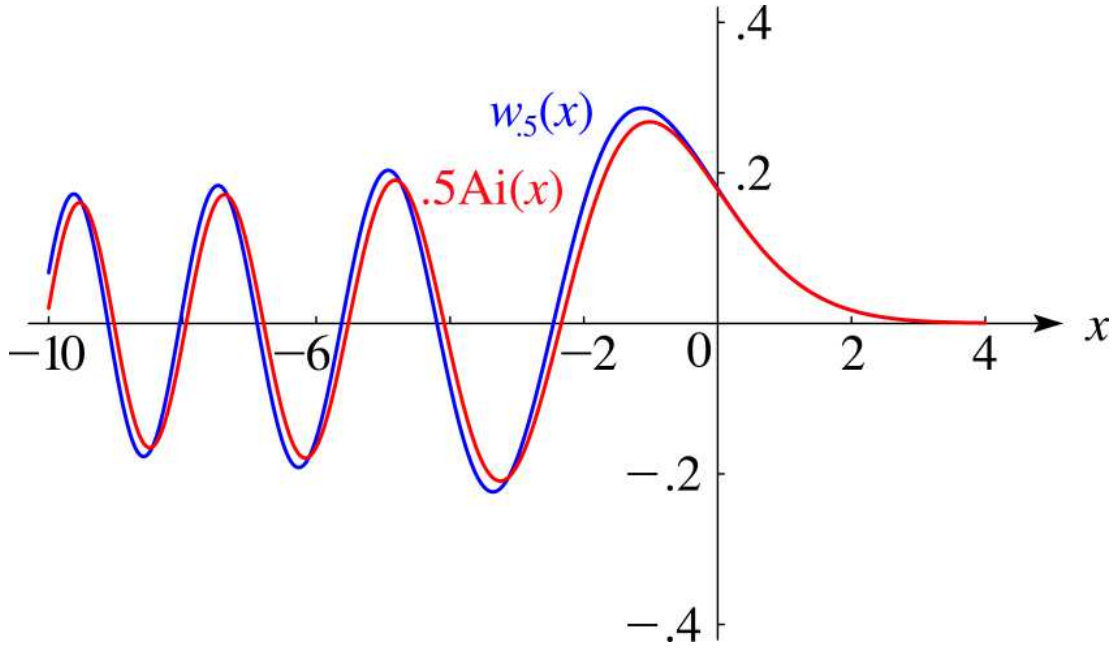


FIGURE 1. Graphs of the Airy function $.5\text{Ai}(x)$ and the nonlinear Airy function $w_{.5} = A_{1/2}(x)$ (red and blue in an electronic version, respectively, [28]).

4.2. Linear Part. Generalized harmonic oscillators [15], [49], [60], [145], [150], which correspond to the Schrödinger equation with variable quadratic Hamiltonians, are well studied in quantum mechanics (see also [34], [35], [50], [83], [88], [133] and references therein for a general approach and known elementary and transcendental solutions). A few examples include the Caldirola–Kanai Hamiltonian of the quantum damped oscillator [19], [43], [72], [141] and some of its natural modifications; a modified oscillator considered by Meiler, Cordero-Soto and Suslov [92], [37], and the degenerate parametric oscillator [38]; the quantum damped oscillator of Chruściński and Jurkowski [26], and a quantum-modified parametric oscillator among others. Green’s functions are derived in a united way in Ref. [36].

4.3. Examples. Combination of linear and nonlinear parts by our formula (2.3) results in numerous explicit soliton-like solutions for corresponding nonlinear Schrödinger equations. It is worth noting that in this approach most linear and some nonlinear classical special functions of mathematical physics are linked together through these solutions.

4.3.1. *Nonlinear Optics.* In the simplest form,

$$i \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + g\psi + h |\psi|^2 \psi, \quad (4.9)$$

one gets [34], [97], [98]

$$\begin{aligned} \alpha(t) &= \frac{\alpha_0}{1 - 4\alpha_0 t}, & \beta(t) &= \frac{\beta_0}{1 - 4\alpha_0 t}, \\ \gamma(t) &= \gamma_0 + \frac{\beta_0^2 t}{1 - 4\alpha_0 t}, & \mu(t) &= \mu_0 (1 - 4\alpha_0 t) \end{aligned} \quad (4.10)$$

($\mu_0, \alpha_0, \beta_0, \gamma_0$ are constants) and

$$z = \frac{\beta_0 x + 2(\gamma_0 + (\beta_0^2 - 4\alpha_0 \gamma_0)t)y}{1 - 4\alpha_0 t}, \quad (4.11)$$

$$g(x, t) = -\frac{g_0 \beta_0^2}{(1 - 4\alpha_0 t)^2} z^m \quad (m = 0, 1), \quad (4.12)$$

$$h(t) = -\frac{h_0 \mu_0 \beta_0^2}{1 - 4\alpha_0 t}. \quad (4.13)$$

(Traditionally, $\alpha_0 = 0$ and $m = 0$ with $\psi = \chi \exp(ig_0 \beta_0^2 t)$ [81], [151]. The case $m = 1$ is discussed in Section 6.)

The case $b = c = 0$,

$$i\psi_t = -a\psi_{xx} - id\psi + g\psi + h|\psi|^2\psi, \quad (4.14)$$

is of interest in fiber optics (see, for example, [4], [7], [61], [77], [78], [79], [97], [98], [116], [119], [123], [124], [126], [127] and references therein). Here, all parameters $a(t)$, $d(t)$ and $h(t)$ are functions of the propagation distance $t = z$ and this equation describes the amplification or attenuation (if d is positive) of pulses propagating nonlinearly in a single-mode optical fiber, where $\psi(t, x)$ is the complex envelope of the electrical field in a comoving frame, $x = \tau$ is the retarded time, $a(t)$ is the group velocity dispersion parameter, $d(t)$ is the dispersion gain or loss function, and $h(t)$ is the nonlinearity parameter [78], [79].

The substitution $\psi = \chi e^{-\Lambda}$, $\Lambda(t) = \int_0^t d(s) ds$ results in

$$i \frac{\partial \chi}{\partial t} = -a \frac{\partial^2 \chi}{\partial x^2} + g\chi + h e^{-2\Lambda} |\chi|^2 \chi, \quad (4.15)$$

which, of course, can be solved by the method under consideration, but a standard change of the time variable,

$$\tau = - \int_0^t a(s) ds, \quad (4.16)$$

transforms this equation into the previous one. Just replace $t \rightarrow \tau$ (see also [78], [79], [123], [124] and [125], where this simple observation has been omitted). More general transformations are discussed in Refs. [2], [27], [35], [82], [98], [112], [116], [134], [155].

4.3.2. *Harmonic Solitons.* In a similar fashion, one can show that the nonlinear Schrödinger equation of the form

$$i\frac{\partial\chi}{\partial t} = \frac{1}{2} \left(-\frac{\partial^2\chi}{\partial x^2} + x^2\chi \right) + \frac{h_0\mu_0\beta_0^2}{2(\cos t + 2\alpha_0 \sin t)} |\chi|^2 \chi \quad (4.17)$$

has the following explicit solution:

$$\chi(x, t) = \frac{e^{iS(x, t)}}{\sqrt{|\mu_0|(\cos t + 2\alpha_0 \sin t)}} F(z), \quad (4.18)$$

where

$$z = \frac{\beta_0 x + (2\gamma_0 \cos t - (\beta_0^2 - 4\alpha_0\gamma_0) \sin t) y}{\cos t + 2\alpha_0 \sin t} \quad (4.19)$$

and

$$\begin{aligned} S(x, t) = & \frac{2\alpha_0 \cos t - \sin t}{2(\cos t + 2\alpha_0 \sin t)} x^2 + \frac{\beta_0 xy}{\cos t + 2\alpha_0 \sin t} \\ & + \frac{2\gamma_0 \cos t - (\beta_0^2 - 4\alpha_0\gamma_0) \sin t}{2(\cos t + 2\alpha_0 \sin t)} y^2 + \frac{g_0\beta_0^2 \sin t}{2(\cos t + 2\alpha_0 \sin t)} \end{aligned} \quad (4.20)$$

provided

$$F'' = g_0 F + h_0 F^3 \quad (4.21)$$

($m = 0$, $\mu_0 \neq 0$, α_0 , β_0 , γ_0 , g_0 , h_0 and y are arbitrary real constants). The reader may choose the profile function F , say in one of the forms (4.1)–(4.4).

If $m = 1$, the Schrödinger equation,

$$\begin{aligned} i\frac{\partial\psi}{\partial t} = & \frac{1}{2} \left(-\frac{\partial^2\psi}{\partial x^2} + x^2\psi \right) + \frac{g_0\beta_0^2}{2(\cos t + 2\alpha_0 \sin t)^2} z\psi \\ & + \frac{h_0\mu_0\beta_0^2}{2(\cos t + 2\alpha_0 \sin t)} |\psi|^2 \psi, \end{aligned} \quad (4.22)$$

has a solution-like solution of the form (4.18) (last term in (4.20) should be omitted) provided that

$$F'' = g_0 z F + h_0 F^3, \quad (4.23)$$

which solution is given in terms of the nonlinear Airy function $A_k(\zeta)$. Further details are left to the reader (see also Section 6).

5. MATTER WAVE SOLITONS

5.1. **Gross–Pitaevskii equation.** Discovery of Bose–Einstein condensates in ultra-cold gases of weakly interacting alkali-metal atoms has stimulated intensive studies of nonlinear matter waves on a macroscopic scale (see, for example, [16], [39], [58], [73], [130]). The Gross–Pitaevskii equation for a zero-temperature condensate of atoms, confined in a cylindrical trap $V_0(x, y) = m\omega_\perp^2(x^2 + y^2)/2$, and a time-dependent harmonic confinement, which can be either attractive or repulsive, along the z direction, $V_1(z, t) = m\omega_0^2(t)z^2/2$, is given by [12], [42], [54], [86], [114], [121], [127]:

$$i\hbar \frac{\partial\Psi(\mathbf{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta + V_{\text{ext}}(\mathbf{r}, t) + U |\Psi(\mathbf{r}, t)|^2 + i\frac{\hbar\eta(t)}{2} \right) \Psi(\mathbf{r}, t), \quad (5.1)$$

where $U = 4\pi\hbar^2 a_s/m$, a_s is the s -wave scattering length, m is the mass of the atom, $V_{\text{ext}} = V_0(x, y) + V_1(z, t)$ and the condensate interaction with the normal atomic cloud through three-body interaction is phenomenologically incorporated by a gain or loss term $\eta(t)$. If the interaction energy of atoms is much less than the kinetic energy in the transverse direction, then the substitution

$$\Psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi a_0 a_\perp}} \exp\left(-i\omega_\perp t - \frac{x^2 + y^2}{2a_\perp^2} + \frac{\Lambda(t)}{2}\right) \times \psi\left(\frac{z}{a_\perp}, \omega_\perp t\right), \quad \Lambda(t) = \int_0^t \eta(s) ds \quad (5.2)$$

allows one to reduce the three-dimensional Gross–Pitaevskii equation (5.1) to the following one-dimensional nonlinear Schrödinger equation in new dimensionless units $\zeta = z/a_\perp$ and $\tau = \omega_\perp t$:

$$i\frac{\partial\psi}{\partial\tau} = \frac{1}{2}\left(-\frac{\partial^2\psi}{\partial\zeta^2} + \omega^2(\tau)\zeta^2\psi\right) + \kappa(\tau)|\psi|^2\psi. \quad (5.3)$$

Here,

$$\kappa(\tau) = 2e^\Lambda \frac{a_s}{a_0}, \quad \omega^2(\tau) = \frac{\omega_0^2}{\omega_\perp^2}, \quad a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}} \quad (5.4)$$

and a_0 is the Bohr radius (see Refs. [68], [76], [93], [99], [100], [111], [121] for more details).

Letting

$$\psi = \chi \exp\left(i \int_0^\tau g(s) ds\right) \quad (5.5)$$

with the function $g(\tau)$ given by (5.9) below, equation (5.3) can be transformed into (2.2), where

$$a = \frac{1}{2}, \quad b = \frac{1}{2}\omega^2(\tau), \quad c = d = 0, \quad h = \kappa(\tau). \quad (5.6)$$

Then our characteristic equation (2.11) take the form

$$\mu'' + \omega^2(\tau)\mu = 0, \quad (5.7)$$

which describes the motion of a classical oscillator with variable frequency [87]. (This equation coincides also with the Ehrenfest theorem for the corresponding linear Hamiltonian [36].) Choosing the standard solutions $\mu_0(\tau)$ and $\mu_1(\tau)$ with $\mu_0(0) = 0$, $\mu'_0(0) = 1$ and $\mu_1(0) \neq 0$, $\mu'_1(0) = 0$, one can use formulas (2.16)–(2.22) with $c = d = 0$ in order to solve the linear problem in quadratures. This gives the soliton travelling wave variable $z = \beta x + 2\gamma y$ and the following balancing conditions:

$$\kappa(\tau) = h_0 \frac{\beta^2(0)\mu^2(0)}{2\mu(\tau)}, \quad (5.8)$$

$$g(\tau) = g_0 \frac{\beta^2(0)\mu^2(0)}{2\mu^2(\tau)}, \quad \int_0^\tau g(s) ds = g_0(\gamma(0) - \gamma(\tau)) \quad (5.9)$$

when $m = 0$.

5.2. Feshbach Resonance. The properties of Bose–Einstein condensed gases can be strongly altered by tuning the external magnetic field. A Feshbach resonance management for Bose–Einstein condensates has been discussed from experimental and theoretical perspectives by many authors (see, for example, [1], [25], [41], [40], [39], [56], [63], [66], [67], [71], [74], [85], [90], [96], [109], [110], [114], [117], [127], [128], [130], [131], [138], [139], [154] and references therein). The Feshbach resonance is a scattering resonance in which pairs of free atoms are tuned via Zeeman effect into resonance with vibrational state of the diatomic molecule [39], [130], [139]. (They are known as Feshbach resonance because of their similarity to scattering resonances described by Herman Feshbach in nuclear collisions.) The strength of the nonlinearity U is defined in terms of s -wave scattering length a_s , namely,

$$U = \frac{4\pi\hbar^2 a_s}{m}, \quad (5.10)$$

and dependence of atomic collision cross section due to existence of the metastable state [39], [56], [71], [128] enables a_s to be continuously tuned from positive to negative values. (The scattering length also determines the formation rate, the spectrum of collective excitations, the evolution of the condensate phase, the coupling with the noncondensed atoms, and other important properties [40], [114].) As follows from the experiments, the s -wave scattering length is the following function of the applied magnetic field [96]:

$$\frac{a_s(B)}{a_0} = a_\infty \left(1 + \frac{\Delta_0}{B_0 - B} \right) \quad (5.11)$$

(The Feshbach resonance provides, so to speak, a continuous knob to adjust the atom-atom interaction from repulsive to attractive, and from weak to strong [130]. Thus it is possible to study strongly interacting, weakly or noninteracting, or collapsing condensates [71], all with the same alkali species and experimental setup. When the nonlinearity $U = 0$, one deals with linear modes of a macroscopic harmonic oscillator [76]; see, for example, [83], [50], [88] and references therein for a detailed treatment of the corresponding quantum oscillator with variable frequency.) In the empirically established expression (5.11), B_0 is the resonant value of the magnetic field, a_∞ is the off-resonance scattering length and parameter Δ_0 represents the resonance width in units of the Bohr radius a_0 (see [39], [96], [127], [138], [139] and references therein for more details). Feshbach resonances have been observed in ^{85}Rb at 164 G [41], [117], [40], in ^{23}Na at 853 and 907 G [67] and have also been identified in ^6Li [66], [106], [129], [130].

5.3. Matter Wave Soliton Management. The Feshbach resonance provides an effective practical tool for experimental study of the matter wave solitons. Indeed, for creation of a certain soliton configuration one needs to satisfy the following condition:

$$h_0 \frac{\beta^2(0) \mu^2(0)}{4\mu} e^{-\Lambda} = a_\infty \left(1 + \frac{\Delta_0}{B_0 - B} \right) \quad (5.12)$$

in order to synchronize the Feshbach resonance and harmonic trap. (Here, both sides have the same simple pole structure, which can be used in experimental setting.) This equation allows determination of the classical law of motion (kinematics in z direction) of the expectation value $\mu = \langle \Psi, \mathbf{r} \Psi \rangle$ with respect to the linear part of Gross–Pitaevskii Hamiltonian (5.1), when $U = \eta = 0$, in terms of a suitable applied magnetic field B near the Feshbach resonance. (The synchronized harmonic trap oscillation frequency should be found from the classical equation of motion (5.7) as

$\omega^2 = -\mu''/\mu$.) Vice versa, the required tuning magnetic field is given by

$$B = B_0 + \frac{4a_\infty \Delta_0 e^\Lambda \mu}{4a_\infty \Delta_0 e^\Lambda \mu - h_0 \beta^2(0) \mu^2(0)}, \quad (5.13)$$

if a particular law of motion μ is obtain by integration (dynamics) of the classical equation (5.7). Our criteria of the wave matter soliton management are consistent with ones obtained in Refs. [12], [63], [126], [127] and [154], if the classical equation of motion (5.7) is taken into account (this point seems not emphasized in these papers).

5.4. Examples. Harmonic matter wave solitons, which correspond to $\omega^2 = \omega_0^2/\omega_\perp^2 = \text{constant}$ in the nonlinear Schrödinger equation (5.3), namely,

$$i \frac{\partial \psi}{\partial \tau} = \frac{1}{2} \left(-\frac{\partial^2 \psi}{\partial \zeta^2} + \omega^2 \zeta^2 \psi \right) + 2 \frac{a_s}{a_0} |\psi|^2 \psi, \quad \eta = 0 \quad (5.14)$$

can be produced in Bose–Einstein condensates by tuning the external magnetic field near the Feshbach resonance as follows

$$B = B_0 + \Delta_0 + \frac{\omega h_0 \mu_0 \beta_0^2 \Delta_0}{4a_\infty (2\alpha_0 \sin \omega \tau + \omega \cos \omega \tau) - \omega h_0 \mu_0 \beta_0^2}. \quad (5.15)$$

Letting $\omega \rightarrow 0$, one gets

$$B = B_0 + \Delta_0 + \frac{h_0 \mu_0 \beta_0^2 \Delta_0}{4a_\infty (2\alpha_0 \tau + 1) - h_0 \mu_0 \beta_0^2}, \quad (5.16)$$

when $\omega_0 = 0$ (a similar case has been recently discussed in Refs. [63] and [127]; it is of interest to analyze possible experimental setup; see also [154]).

The reader may find more details on the synchronization of Feshbach resonance and harmonic trap, explicit soliton configurations, and available numerical and experimental results in recent papers [12], [63], [126], [127], [148] (see also [9], [10], [17], [52], [78], [79], [140], [149], [151] and references therein).

6. GENERALIZATION

If an arbitrary linear combination of operators $p = -i\partial/\partial x$ and x is added to the quadratic Hamiltonian in equation (2.2), namely,

$$i\psi_t = -a(t) \psi_{xx} + b(t) x^2 \psi - ic(t) x \psi_x - id(t) \psi - f(t) x \psi + ig(t) \psi_x + h(t) |\psi|^2 \psi, \quad (6.1)$$

one can look for exact solutions in a more general form

$$\begin{aligned} \psi &= A(x, t) e^{iS(x, t)}, \\ S(x, t) &= \alpha(t) x^2 + \beta(t) xy + \gamma(t) y^2 + \delta(t) x + \varepsilon(t) y + \kappa(t) + \xi(t) \end{aligned} \quad (6.2)$$

(y is a parameter, we are separating contributions from linear κ and nonlinear ξ parts in the constant term). The linear part has been already solved in [34] and [132]. One has additional equations

$$\frac{d\delta}{dt} + (c + 4a\alpha) \delta = f + 2\alpha g, \quad (6.3)$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta) \beta, \quad (6.4)$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 \quad (6.5)$$

to the system (2.7)–(2.9), whose solutions are given by

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (6.6)$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (6.7)$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))}. \quad (6.8)$$

Here,

$$\delta_0(t) = \frac{\lambda(t)}{\mu_0(t)} \int_0^t \left[\left(f(s) - \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu'_0(s) \right] \frac{ds}{\lambda(s)}, \quad (6.9)$$

$$\begin{aligned} \varepsilon_0(t) = & -\frac{2a(t)\lambda(t)}{\mu'_0(t)} \delta_0(t) + 8 \int_0^t \frac{a(s)\sigma(s)\lambda(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s)) ds \\ & + 2 \int_0^t \frac{a(s)\lambda(s)}{\mu'_0(s)} \left(f(s) - \frac{d(s)}{a(s)} g(s) \right) ds, \end{aligned} \quad (6.10)$$

$$\begin{aligned} \kappa_0(t) = & \frac{a(t)\mu_0(t)}{\mu'_0(t)} \delta_0^2(t) - 4 \int_0^t \frac{a(s)\sigma(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s))^2 ds \\ & - 2 \int_0^t \frac{a(s)}{\mu'_0(s)} (\mu_0(s)\delta_0(s)) \left(f(s) - \frac{d(s)}{a(s)} g(s) \right) ds \end{aligned} \quad (6.11)$$

with $\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$ and $\kappa_0(0) = 0$ (see Refs. [34] and [132] for more details).

Our equation (3.2) takes the form

$$A_t + ((4a\alpha + c)x + 2a\beta y + 2\delta a - g)A_x + (2\alpha a + d)A = 0 \quad (6.12)$$

and the nonlinear equation (3.3) becomes

$$aA_{xx} = \frac{d\xi}{dt}A + hA^3. \quad (6.13)$$

Once again, with the help of (2.8), (2.10) and (6.4), equation (6.12) can be rewritten as

$$A_t - \left(\frac{\beta'}{\beta}x - 2a\beta y + \frac{\varepsilon'}{\beta} \right) A_x + \frac{1}{2} \frac{\mu'}{\mu} A = 0 \quad (6.14)$$

and looking for a travelling wave solution of the form

$$A = A(x, t) = \frac{1}{\sqrt{\mu(t)}} F(z), \quad z = c_0(t)x + c_1(t)y + c_2(t), \quad (6.15)$$

one gets

$$c'_0x + c'_1y + c'_2 = \left(\frac{\beta'}{\beta}x - 2a\beta y + \frac{\varepsilon'}{\beta} \right) c_0 \quad (6.16)$$

with $c_0 = \beta$, $c_1 = 2\gamma$ and $c_2 = \varepsilon$ (or $z = \beta x + 2\gamma y + \varepsilon$). Then

$$\frac{d^2}{dz^2} F(z) = \frac{d\xi/dt}{a\beta^2} F(z) + \frac{h}{a\beta^2\mu} F^3(z) \quad (6.17)$$

and our balancing conditions are given by

$$\frac{d\xi}{dt} = g_0 a(t) \beta^2(t), \quad h = h_0 a(t) \beta^2(t) \mu(t) = h_0 \beta^2(0) \mu^2(0) \frac{a(t) \lambda^2(t)}{\mu(t)} \quad (6.18)$$

with $\xi = g_0 (\gamma(0) - \gamma(t))$ according to (2.9). For the soliton velocity,

$$x' + \frac{\beta'}{\beta} x = 2a\beta y + 2a\delta - g, \quad (6.19)$$

thus extending (2.13). Equation of motion is given by

$$\begin{aligned} x'' - \frac{a'}{a} x' + \left(4ab - c^2 + c \left(\frac{a'}{a} - \frac{c'}{c} \right) \right) x \\ = 2af + \left(\frac{a'}{a} - c \right) g - g' \end{aligned} \quad (6.20)$$

as a nonhomogeneous generalization of (2.15) (forced damped parametric oscillators).

As a final result, our solitary wave solution has the form

$$\begin{aligned} \psi(x, t) &= \frac{e^{i\phi}}{\sqrt{\mu}} \exp \left(i \left(\alpha x^2 + \beta xy + \gamma (y^2 - g_0) + \delta x + \varepsilon y + \kappa \right) \right) \\ &\times F(\beta x + 2\gamma y + \varepsilon), \end{aligned} \quad (6.21)$$

where the elliptic function F satisfies the nonlinear equation (2.5) with $m = 0$ and ϕ, y, g_0 and h_0 are real parameters. Time-dependent functions $\mu(t), \alpha(t), \beta(t), \gamma(t), \delta(t)$ and $\kappa(t)$ are given by our equations (2.16)–(2.22) and (6.6)–(6.11) (as in the corresponding solution of the linear Schrödinger equation [34], [132]).

Example 1. A soliton motion with acceleration in linearly inhomogeneous plasma was discovered in Refs. [22] and [23] (see also [13], [137]). For a modified equation,

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2kx\psi = \frac{h_0 \mu_0 \beta_0^2}{1 + 4\alpha_0 t} |\psi|^2 \psi, \quad (6.22)$$

where $k, h_0, \alpha_0, \beta_0$ and μ_0 are constants, we get $\mu(t) = \mu_0 (1 + 4\alpha_0 t)$ and

$$\begin{aligned} \alpha(t) &= \frac{\alpha_0}{1 + 4\alpha_0 t}, & \beta(t) &= \frac{\beta_0}{1 + 4\alpha_0 t}, \\ \gamma(t) &= \gamma_0 - \frac{\beta_0^2 t}{1 + 4\alpha_0 t}, & \delta(t) &= kt + \frac{\delta_0 + kt}{1 + 4\alpha_0 t}, \\ \varepsilon(t) &= \varepsilon_0 - \frac{2\beta_0 t (\delta_0 + kt)}{1 + 4\alpha_0 t}, & \kappa(t) &= \kappa_0 - \frac{k^2 t^3}{3} - \frac{t (\delta_0 + kt)^2}{1 + 4\alpha_0 t} \end{aligned} \quad (6.23)$$

with

$$\begin{aligned} z &= \beta x + 2\gamma y + \varepsilon \\ &= \beta_0 \frac{x - 2t(\beta_0 y + \delta_0 + kt)}{1 + 4\alpha_0 t} + 2y\gamma_0 + \varepsilon_0 \end{aligned} \quad (6.24)$$

in our particular solution (6.21). The classical case [22], [23] corresponds to $\alpha_0 = 0$ and $h_0 \mu_0 \beta_0^2 = -2$ (with $k \rightarrow -k$). The reader may choose the profile F in one of the forms (4.1)–(4.4).

Example 2. A similar case occur, if one takes $m = 1$ in (4.12). The corresponding Schrödinger equation is given by

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} = \frac{g_0\beta_0^2}{(1+4\alpha_0 t)^2} z\psi + \frac{h_0\mu_0\beta_0^2}{1+4\alpha_0 t} |\psi|^2 \psi, \quad (6.25)$$

where

$$z = \frac{\beta_0 x + 2(\gamma_0 - (\beta_0^2 - 4\alpha_0\gamma_0)t)y}{1+4\alpha_0 t}. \quad (6.26)$$

With the help of the gauge transformation,

$$\psi = e^{-if(t)}\chi(x, t), \quad \frac{df}{dt} = 2g_0\beta_0^2 y \frac{\gamma_0 - (\beta_0^2 - 4\alpha_0\gamma_0)t}{(1+4\alpha_0 t)^3}, \quad (6.27)$$

one gets

$$i\chi_t + \chi_{xx} - \frac{g_0\beta_0^3 x}{(1+4\alpha_0 t)^3}\chi = \frac{h_0\mu_0\beta_0^2}{1+4\alpha_0 t} |\chi|^2 \chi \quad (6.28)$$

and

$$\chi(x, t) = \frac{e^{iS(x, t)}}{\sqrt{|\mu_0|(1+4\alpha_0 t)}} g_0^{1/3} \sqrt{\frac{2}{h_0}} A_k \left(g_0^{1/3} z \right). \quad (6.29)$$

Here,

$$S(x, t) = \frac{\alpha_0 x^2 + \beta_0 xy + (\gamma_0 - (\beta_0^2 - 4\alpha_0\gamma_0)t)y^2}{1+4\alpha_0 t} + g_0\beta_0^2 t \frac{2\gamma_0 - (\beta_0^2 - 8\alpha_0\gamma_0)t}{1+4\alpha_0 t} y \quad (6.30)$$

and the soliton profile is defined (as a solution of the second Painlevé equation) in terms of the nonlinear Airy function $A_k(\zeta)$ with asymptotics given by (4.6)–(4.8). (Graph of one of these functions, $w_5 = A_{1/2}$, is presented on Figure 1 from [28].) It is worth noting that, in a contrast to the previous case, our A -soliton moves with a constant velocity when $\alpha_0 = 0$. Further details are left to the reader.

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